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A CUSUM of squares test for cointegration using OLS regression residuals

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- Since the work by Engle and Granger (EG) (1987), and with time series data, there has been a growing interest and use in modelling, implications and treatment of the concept of **cointegration** when dealing with highly persistent data in regression and VAR-type models.
- Some recent and relevant extensions:
    - Fractional cointegration and unbalanced cointegrating regressions
    - Co-summability
    - Time-varying cointegration
    - Functional-coefficient cointegration
    - Threshold cointegration and cointegration with threshold effects
- A great amount of research has been devoted to the crucial question of testing for the existence of a certain number of stationary (linear) combinations of the non-stationary system variables, that is, to testing for the existence of cointegrating relationships, also known as stable long-run relationships.
- There are three main approaches to testing for cointegration: **(1)** a system based FIML estimation of a VECM (Johansen (1989)), **(2)** a two-step procedure based on a single-equation regression (EG), and **(3)** a single-equation conditional ECM (Banerjee et.al. (1986)). Zivot (2000) discusses various examples for which economic theory can imply a single cointegrating vector and explains why it is reasonable to test for cointegration in single equation regression models in such cases.
- In the context of a single-equation regression model, there are mainly two types of testing procedures for cointegration: **(a)** T-type test statistics (DF/ADF or PP) for the null hypothesis of no cointegration based on an auxilary regression for the OLS residuals, and **(b) Fluctuation-type test statistics** for testing the null of cointegration based on regression residuals (OLS/FM-OLS/CCR/ DOLS).
- This work contribute to this last second family of testing procedures for the null of cointegration with a relatively simple to compute and easy to use new fluctuation-type test statistic.

A CUSUM of squares test for cointegration using OLS regression residuals

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1. The model and basic assumptions

The set of k+1 observable integrated of order 1, I(1), series admits the decomposition

$$\begin{pmatrix} Y_t \\ \mathbf{X}_{k,t} \end{pmatrix} = \begin{pmatrix} d_{0,t} \\ \mathbf{d}_{k,t} \end{pmatrix} + \begin{pmatrix} \eta_{0,t} \\ \boldsymbol{\eta}_{k,t} \end{pmatrix} \quad t = 1,\dots,n$$

where $\mathbf{d}_t$ Is the underlying deterministic component given by a polynomial trend function, such as

$$\mathbf{d}_t = \begin{pmatrix} d_{0,t} \\ \mathbf{d}_{k,t} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}'_{0,p}\boldsymbol{\tau}_{p,t} \\ \mathbf{A}_{k,p}\boldsymbol{\tau}_{p,t} \end{pmatrix}, \quad \boldsymbol{\tau}_{p,t} = (1, t, \dots, t^p)', p \geq 0$$

and $\boldsymbol{\eta}_t$ is the stochastic trend component defined as

$$\boldsymbol{\eta}_t = \begin{pmatrix} \eta_{0,t} \\ \boldsymbol{\eta}_{k,t} \end{pmatrix} = \boldsymbol{\eta}_{t-1} + \boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_t = \begin{pmatrix} \varepsilon_{0,t} \\ \boldsymbol{\varepsilon}_{k,t} \end{pmatrix}$$

Under conditions ensuring a multivariate invariance principle for the zero mean $\boldsymbol{\varepsilon}_t$ sequence

$$(1/\sqrt{n})\sum_{t=1}^{[nr]} \boldsymbol{\varepsilon}_t = (1/\sqrt{n})\sum_{t=1}^{[nr]} \begin{pmatrix} \varepsilon_{0,t} \\ \boldsymbol{\varepsilon}_{k,t} \end{pmatrix} \Rightarrow \mathbf{C}(r) = \begin{pmatrix} C_0(r) \\ \mathbf{C}_k(r) \end{pmatrix} = \mathbf{BM}(\boldsymbol{\Gamma}), \boldsymbol{\Gamma} = \begin{pmatrix} \gamma_0^2 & \boldsymbol{\gamma}_{0k} \\ \boldsymbol{\gamma}_{k0} & \boldsymbol{\Gamma}_{kk} \end{pmatrix}, \boldsymbol{\Gamma}_{kk} > 0$$

if there exists a k-vector $\boldsymbol{\beta}_k$ such that $u_t = (1, -\boldsymbol{\beta}'_k)\begin{pmatrix} \eta_{0,t} \\ \boldsymbol{\eta}_{k,t} \end{pmatrix} = \mathbf{c}'_k\boldsymbol{\eta}_t$ is stationary, then the underlying stochastic trend components (and hence the observed series) are cointegrated in the sense of Engle and Granger (1987), and the long-run covariance matrix $\boldsymbol{\Gamma}$ is singular. These results justify the specification of the (static and linear) **cointegrating regression equation model** given by

$$Y_t = \boldsymbol{\alpha}'_p\boldsymbol{\tau}_{p,t} + \boldsymbol{\beta}'_k\mathbf{X}_{k,t} + u_t \quad t = 1,\dots,n \qquad \boldsymbol{\alpha}_p = \boldsymbol{\alpha}_{0,p} - \mathbf{A}'_{k,p}\boldsymbol{\beta}_k$$

1. The model and basic assumptions

Given the specification of the cointegrating regression equation model

$$Y_t = \boldsymbol{\alpha}_p' \boldsymbol{\tau}_{p,t} + \boldsymbol{\beta}_k' \mathbf{X}_{k,t} + u_t \quad t = 1,\ldots,n$$

with a set of p+1 ≥ 1 (p ≥ 0) deterministic trending regressors and k ≥ 1 stochastic trending integrated regressors, the relevant asymptotic results are based on the following explicit assumption concerning the behavior of the error terms.

Assumption L. (Linear process)

(*a*) The regression error is given by $u_t = \alpha u_{t-1} + \upsilon_t$, with $0 \leq \alpha \leq 1$, and

(*b*) $\boldsymbol{\xi}_t = (\upsilon_t, \boldsymbol{\varepsilon}_{k,t}')'$, with $\boldsymbol{\xi}_t = \mathbf{D}(L)\mathbf{e}_t$, where $\mathbf{e}_t = (e_{0,t}, \mathbf{e}_{k,t}')'$ is a k+1-variate iid process with zero mean, covariance matrix $\boldsymbol{\Sigma}_e > 0$ and mth-order finite moment, $E[|\mathbf{e}_t|^m] < \infty$ for some $m \geq 2$. Also, for the infinite order polynomial matrix in the lag operator L, $\mathbf{D}(L) = (\mathbf{d}_0(L), \mathbf{D}_k'(L))' = \sum_{j=0}^{\infty} \mathbf{D}_j L^j$, it is assumed that $|\mathbf{D}(1)| = \mathrm{Det}(\mathbf{D}(1)) \neq 0$ (nonsingular), with coefficients satisfying the summability condition $\sum_{j=1}^{\infty} j^2 \|\mathbf{D}_j\| < \infty$, with $\|\cdot\|$ a matrix norm defined as $\|\mathbf{D}_j\| = [Tr(\mathbf{D}_j'\mathbf{D}_j)]^{1/2}$.

- $$(1/\sqrt{n}) \sum_{t=1}^{[nr]} \boldsymbol{\xi}_t = \mathbf{D}(1)(1/\sqrt{n}) \sum_{t=1}^{[nr]} \mathbf{e}_t + O_p(n^{-1/2}) \Rightarrow \mathbf{B}(r) = \begin{pmatrix} B_\upsilon(r) \\ \mathbf{B}_k(r) \end{pmatrix} = \mathbf{BM}(\boldsymbol{\Omega}), \boldsymbol{\Omega} = \mathbf{D}(1)\Sigma_e \mathbf{D}(1)'$$
- For $0 \leq \alpha < 1$:

$$n^{-1/2} \sum_{t=1}^{[nr]} \begin{pmatrix} u_t \\ \boldsymbol{\varepsilon}_{k,t} \end{pmatrix} \Rightarrow \begin{pmatrix} B_u(r) \\ \mathbf{B}_k(r) \end{pmatrix} = \mathbf{BM}(\boldsymbol{\Omega}_u) = \begin{pmatrix} (1-\alpha)^{-1} B_\upsilon(r) \\ \mathbf{B}_k(r) \end{pmatrix}, B_u(r) = \omega_{u.k} W_u(r) + \boldsymbol{\omega}_{uk} \boldsymbol{\Omega}_{kk}^{-1} \mathbf{B}_k(r)$$

$$\boldsymbol{\Omega}_u = \begin{pmatrix} \omega_u^2 & \boldsymbol{\omega}_{uk} \\ \boldsymbol{\omega}_{ku} & \boldsymbol{\Omega}_{kk} \end{pmatrix} = \mathbf{C}(1)\boldsymbol{\Sigma}_e \mathbf{C}'(1), \mathbf{C}(L) = \begin{pmatrix} (1-\alpha L)^{-1} \mathbf{d}_0'(L) \\ \mathbf{D}_k(L) \end{pmatrix}$$

1. The model and basic assumptions

A note on a more general family of cointegrating regression models where the proposed testing procedure is also of application: **Functional-coefficient regression**

$$Y_t = \alpha'_p(Z_t)\tau_{p,t} + \beta'_k(Z_t)\mathbf{X}_{k,t} + u_t \quad t = 1,\ldots,n$$

Z_t : functional variable (univariate)

		X's	Z	u
Cai, Li and Park (2009). Local-linear kernel estimator	(a)	I(0)/I(1)	I(0)	I(0)
	(b)	I(0)	I(1)	I(0)
Xiao (2009). Kernel estimator		I(1)	I(0)	I(0)
Sun, Hsiao, and Li (2011). OLS/Local constant kernel estimator		I(1)	I(0)	I(1)
Pitarakis (2012). Piecewise local linear estimator		I(1)	I(0)	I(1)
Sun, Cai and Li (2013). Local-linear kernel estimator		I(1)	I(1)	I(0)

$$\theta(Z_t) = \theta(z) + \theta^{(1)}(z)(Z_t - z), \quad \theta^{(s)}(z) = \frac{\partial^s \theta(z)}{\partial z^s}$$

Particular cases: (1) Deterministic structural breaks/deterministic time-varying coeffs.

(2) Cointegration with threshold effects (Gonzalo/Pitarakis (2006):

$$\theta(Z_t) = \theta + \lambda I(q_{t-d} > \gamma)$$

Consistent estimation: OLS, IM-OLS (Afonso-Rodríguez (2013))

2. OLS estimation and residual-based tests for cointegration

Asymptotics (Ref.: Hansen (1992))

$$\mathbf{m}_t = \begin{pmatrix} \boldsymbol{\tau}_{p,t} \\ \mathbf{X}_{k,t} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Gamma}_{p,n}^{-1} & \mathbf{0}_{p+1,k} \\ \mathbf{A}_{k,p}\boldsymbol{\Gamma}_{p,n}^{-1} & \sqrt{n}\mathbf{I}_{k,k} \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau}_{p,tn} \\ \boldsymbol{\eta}_{k,tn} \end{pmatrix} = \mathbf{W}_n \mathbf{m}_{t,n}$$

$$\boldsymbol{\tau}_{p,tn} = \boldsymbol{\Gamma}_{p,n}\boldsymbol{\tau}_{p,t}$$
$$\boldsymbol{\Gamma}_{p,n} = \mathrm{diag}(1, n^{-1}, ..., n^{-p})$$
$$\boldsymbol{\eta}_{k,tn} = n^{-1/2}\boldsymbol{\eta}_{k,t}$$

$$\mathbf{m}_{[nr],n} \Rightarrow \mathbf{m}(r) = \begin{pmatrix} \boldsymbol{\tau}_p(r) \\ \mathbf{B}_k(r) \end{pmatrix}: \quad (1/n)\sum_{t=1}^{n} \mathbf{m}_{t,n}\mathbf{m}'_{t,n} \Rightarrow \int_0^1 \mathbf{m}(r)\mathbf{m}'(r)dr > 0 \text{ a.s. (full - ranked process)}$$

OLS estimator:

$$\begin{pmatrix} \hat{\boldsymbol{\alpha}}_{p,n} \\ \hat{\boldsymbol{\beta}}_{k,n} \end{pmatrix} = \left(\sum_{t=1}^{n} \mathbf{m}_t \mathbf{m}'_t \right)^{-1} \sum_{t=1}^{n} \mathbf{m}_t Y_t = \begin{pmatrix} \boldsymbol{\alpha}_p \\ \boldsymbol{\beta}_k \end{pmatrix} + n^{-v}(\mathbf{W}'_n)^{-1} \left((1/n)\sum_{t=1}^{n} \mathbf{m}_{t,n}\mathbf{m}'_{t,n} \right)^{-1} n^{-(1-v)} \sum_{t=1}^{n} \mathbf{m}_{t,n} u_t$$

Scaled and weighted OLS estimation errors:

$$\hat{\boldsymbol{\Theta}}_{pk,n} = n^v \mathbf{W}'_n \begin{pmatrix} \hat{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p \\ \hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k \end{pmatrix} = \begin{pmatrix} n^v \boldsymbol{\Gamma}_{p,n}^{-1}[(\hat{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p) + \mathbf{A}'_{k,p}(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k)] \\ n^{1/2+v}(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) \end{pmatrix}$$

$$= \left((1/n)\sum_{t=1}^{n} \mathbf{m}_{t,n}\mathbf{m}'_{t,n} \right)^{-1} n^{-(1-v)} \sum_{t=1}^{n} \mathbf{m}_{t,n} u_t \underset{\substack{(v=1/2) \\ \text{Cointegration}}}{\Rightarrow} \underbrace{\left(\int_0^1 \mathbf{m}(s)\mathbf{m}'(s)ds \right)^{-1} \left(\int_0^1 \mathbf{m}(s)dB_u(s) + \begin{pmatrix} \mathbf{0}_{p+1} \\ \boldsymbol{\Delta}_{ku} \end{pmatrix} \right)}_{\text{Mean and covariance matrix mixture of non-Gaussian distributions}}$$

Also : $\hat{\boldsymbol{\beta}}_{k,n} \to^p \boldsymbol{\beta}_k = \boldsymbol{\Omega}_{kk}^{-1}\boldsymbol{\omega}_{ku}$ (superconsistent)

$$\underset{\substack{(v=-1/2) \\ \text{No cointegration}}}{\Rightarrow} \underbrace{\left(\int_0^1 \mathbf{m}(s)\mathbf{m}'(s)ds \right)^{-1} \int_0^1 \mathbf{m}(s)B_u(s)ds}_{\text{Scale mixture of normals centered at } \boldsymbol{\Omega}_{kk}^{-1}\boldsymbol{\omega}_{k\upsilon} \text{ (Phillips (1989))}}$$

2. OLS estimation and residual-based tests for cointegration

Given the sequence of OLS residuals

$$\hat{u}_{t,p}(k) = Y_t - \boldsymbol{\tau}'_{p,t}\hat{\boldsymbol{\alpha}}_{p,n} - \mathbf{X}'_{k,t}\hat{\boldsymbol{\beta}}_{k,n} = u_t - \mathbf{m}'_t \begin{pmatrix} \hat{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p \\ \hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k \end{pmatrix} = u_t - n^{-v}\mathbf{m}'_{t,n}\hat{\boldsymbol{\Theta}}_{pk,n}$$

the basic component is the partial sum process given by

$\hat{U}_{t,p}(k) = \sum_{j=1}^{t} \hat{u}_{j,p}(k) = O_p(n^{1/2})$ under the cointegration assumption

$= O_p(n^{3/2})$ under no cointegration

Fluctuation-type test statistics:

(1) Quadratic-total variation (QvM metric) (Shin(1994), Harris & Inder(1994), Leybourne & McCabe(1994))

$$C\hat{I}_{n,p}(k) = \frac{1}{n^2 \hat{\omega}^2_{u.k,n}(m_n)} \sum_{t=1}^{n} \hat{U}^2_{t,p}(k)$$

(2) Maximum variation (KS metric) (Xiao (1999), Wu and Xiao (2008))

$$\hat{R}_{n,p}(k) = \frac{1}{\sqrt{n}\hat{\omega}_{u.k,n}(m_n)} \max_{t=1,\ldots,n} |\hat{U}_{t,p}(k) - (t/n)\hat{U}_{n,p}(k)|$$

$$C\hat{S}_{n,p}(k) = \frac{1}{\sqrt{n}\hat{\omega}_{u.k,n}(m_n)} \max_{t=1,\ldots,n} |\hat{U}_{t,p}(k)|$$

Except under serially uncorrelated regression errors and weakly exogeneous regressors, their limiting null distributions, that depend on the deterministic components and number of integrated regressors, do not allow for valid inference

with $\hat{\omega}^2_{u.k,n}(m_n)$ a consistent estimator under cointegration of the conditional long-run variance $\omega^2_{u.k}$:
(a) plug-in estimator based on OLS residuals, (b) based on FM-OLS/CCR/DOLS residuals

Other similar proposals:

Hansen (1992), Jansson (2005) (PO test), McCabe, Leybourne and Shin (1997), Stock (1999)

3. A new CUSUM-type testing procedure for the null of cointegration

Motivation. (From Xiao and Lima (2007), “Testing covariance stationarity”)

$$n^{-1/2}\sum_{t=1}^{[nr]}\begin{pmatrix}u_t\\ v_t\end{pmatrix}=\begin{pmatrix}n^{-1/2}\sum_{t=1}^{[nr]}u_t\\ n^{-1/2}\sum_{t=1}^{[nr]}(u_t^2-\sigma_n^2)\end{pmatrix},\quad \sigma_n^2=(1/n)\sum_{j=1}^{n}u_j^2$$

Under stationarity (i.e., under cointegration), the scaled partial sum of centered squared regression errors admits the decomposition:

$$n^{-1/2}\sum_{t=1}^{[nr]}v_t=n^{-1/2}\sum_{t=1}^{[nr]}(u_t^2-\sigma_u^2)-\frac{[nr]}{n}n^{-1/2}\sum_{t=1}^{n}(u_t^2-\sigma_u^2)$$

Under no stationarity (i.e., under no cointegration), we have that:

$$n^{-5/2}\sum_{t=1}^{[nr]}\begin{pmatrix}u_t\\ v_t\end{pmatrix}=\begin{pmatrix}(1/n)\left\{(1/n)\sum_{t=1}^{[nr]}(u_t/\sqrt{n})\right\}\\ (1/n)\sum_{t=1}^{[nr]}[(u_t/\sqrt{n})^2-(1/n)\sigma_n^2]\end{pmatrix}=\begin{pmatrix}O_p(n^{-1})\\ (1/n)\sum_{t=1}^{[nr]}[(u_t/\sqrt{n})^2-(1/n)\sigma_n^2]\end{pmatrix}$$

so that the behavior under no stationarity is dominated by the second component, reflecting the violation of the covariance stationarity assumption induced by the unit root.

Some other alternative testing procedures based on the squared OLS residuals from a cointegrating regression:

- Lu, Maekawa, and Lee (2008) (CUSUM of squares test for structural stability in ARX(1) with integrated regressors)
- Xiao (2009)

3. A new CUSUM-type testing procedure for the null of cointegration

The scaled partial sum of squared and centered OLS residuals from the estimation of a (linear) cointegrating regression model:

$$(1/\sqrt{n})\sum_{t=1}^{[nr]}\hat{v}_{t,p}(k) = (1/\sqrt{n})\sum_{t=1}^{[nr]}(\hat{u}_{t,p}^{2}(k) - \hat{\sigma}_{n,p}^{2}(k)) \qquad \hat{\sigma}_{n,p}^{2}(k) = (1/n)\sum_{j=1}^{n}\hat{u}_{j,p}^{2}(k)$$

$$= (1/\sqrt{n})\sum_{t=1}^{[nr]} v_t + n^{1/2-2v}\hat{\boldsymbol{\Theta}}'_{pk,n}\left(n^{-1}\sum_{t=1}^{[nr]}\mathbf{m}_{t,n}\mathbf{m}'_{t,n} - \frac{[nr]}{n}n^{-1}\sum_{t=1}^{n}\mathbf{m}_{t,n}\mathbf{m}'_{t,n}\right)\hat{\boldsymbol{\Theta}}_{pk,n}$$

$$-2n^{1/2-2v}\hat{\boldsymbol{\Theta}}'_{pk,n}\left(n^{-(1-v)}\sum_{t=1}^{[nr]}\mathbf{m}_{t,n}u_t - \frac{[nr]}{n}n^{-(1-v)}\sum_{t=1}^{n}\mathbf{m}_{t,n}u_t\right)$$

Under the linear process assumption on the error terms, $m > 4$, and under cointegration:

$$(1/\sqrt{n})\sum_{t=1}^{[nr]}\begin{pmatrix} u_t \\ v_t \\ \boldsymbol{\varepsilon}_{k,t}\end{pmatrix} \Rightarrow \begin{pmatrix} B_u(r) \\ B_v(r) \\ \mathbf{B}_k(r)\end{pmatrix}, \quad B_v(r) = \omega_{v.k}W_v(r) + \boldsymbol{\gamma}'_{kv}\mathbf{B}_k(r), \quad \boldsymbol{\gamma}_{kv} = \boldsymbol{\Omega}_{kk}^{-1}\boldsymbol{\omega}_{kv}$$

$$(1/\sqrt{n})\sum_{t=1}^{[nr]}\hat{v}_{t,p}(k) = (1/\sqrt{n})\sum_{t=1}^{[nr]}v_t + O_p(n^{-1/2}) \Rightarrow B_v(r) - rB_v(1)$$

$$= \omega_{v.k}(W_v(r) - rW_v(1)) + \boldsymbol{\gamma}'_{kv}(\mathbf{B}_k(r) - r\mathbf{B}_k(1))$$

(modified) CUSUM of squared OLS residuals:

$$\hat{K}_{n,p}(k) = \frac{1}{\sqrt{n}\hat{\omega}_{v.k,n}(m_n)}\max_{t=1,\dots,n}\left|\sum_{j=1}^{t}\hat{v}_{j,p}(k) - \hat{\boldsymbol{\gamma}}'_{kv,n}(m_n)(\mathbf{X}_{k,t} - (t/n)\mathbf{X}_{k,n})\right|$$

$$\hat{K}_{n,p}(k) \Rightarrow \sup_{r\in[0,1]}|W_v(r) - rW_v(1)|$$ (supremum of a standard Brownian Bridge)

3. A new CUSUM-type testing procedure for the null of cointegration

A note. Scaled partial sum of squared and centered **FM-OLS residuals** (Phillips and Hansen (1990)) from the estimation of a (linear) cointegrating regression model

Fully-modified (FM)-OLS estimator:

$$\begin{pmatrix} \hat{\boldsymbol{\alpha}}^{+}_{p,n} \\ \hat{\boldsymbol{\beta}}^{+}_{k,n} \end{pmatrix} = \left(\sum_{t=1}^{n} \mathbf{m}_t \mathbf{m}_t'\right)^{-1} \left(\sum_{t=1}^{n} \mathbf{m}_t Y_t^{+} - n \begin{pmatrix} 0_{p+1} \\ \hat{\boldsymbol{\Delta}}^{+}_{ku,n}(m_n) \end{pmatrix}\right), \quad Y_t^{+} = Y_t - \hat{\boldsymbol{\gamma}}'_{ku,n}(m_n)\hat{\mathbf{Z}}_{kt,p}$$

$$\begin{aligned}
(1/\sqrt{n})\sum_{t=1}^{[nr]} \hat{v}^{+}_{t,p}(k) &= (1/\sqrt{n})\sum_{t=1}^{[nr]} (z_t^2 - \sigma_n^2) \qquad\qquad z_t = u_t - \boldsymbol{\gamma}'_{ku}\boldsymbol{\varepsilon}_{k,t} \\
&+ (\hat{\boldsymbol{\gamma}}_{ku,n} - \boldsymbol{\gamma}_{ku})' \left\{ n^{-1/2}\sum_{t=1}^{[nr]} (\boldsymbol{\varepsilon}_{k,t}\boldsymbol{\varepsilon}'_{k,t} - \boldsymbol{\Sigma}_{kk}) - \frac{[nr]}{n} n^{-1/2}\sum_{t=1}^{n} (\boldsymbol{\varepsilon}_{k,t}\boldsymbol{\varepsilon}'_{k,t} - \boldsymbol{\Sigma}_{kk}) \right\} (\hat{\boldsymbol{\gamma}}_{ku,n} - \boldsymbol{\gamma}_{ku}) \\
&- n^{1/2-2v} \hat{\boldsymbol{\Theta}}'^{+}_{pk,n} \left(n^{-1}\sum_{t=1}^{[nr]} \mathbf{m}_{t,n}\mathbf{m}'_{t,n} - \frac{[nr]}{n} n^{-1}\sum_{t=1}^{n} \mathbf{m}_{t,n}\mathbf{m}'_{t,n} \right) \hat{\boldsymbol{\Theta}}^{+}_{pk,n} \\
&- 2(\hat{\boldsymbol{\gamma}}_{ku,n} - \boldsymbol{\gamma}_{ku})' \left\{ n^{-1/2}\sum_{t=1}^{[nr]} (\boldsymbol{\varepsilon}_{k,t}u_t - \boldsymbol{\sigma}_{ku}) - \frac{[nr]}{n} n^{-1/2}\sum_{t=1}^{n} (\boldsymbol{\varepsilon}_{k,t}u_t - \boldsymbol{\sigma}_{ku}) \right\} \\
&+ 2(\hat{\boldsymbol{\gamma}}_{ku,n} - \boldsymbol{\gamma}_{ku})' \left\{ n^{-1/2}\sum_{t=1}^{[nr]} (\boldsymbol{\varepsilon}_{k,t}\boldsymbol{\varepsilon}'_{k,t} - \boldsymbol{\Sigma}_{kk}) - \frac{[nr]}{n} n^{-1/2}\sum_{t=1}^{n} (\boldsymbol{\varepsilon}_{k,t}\boldsymbol{\varepsilon}'_{k,t} - \boldsymbol{\Sigma}_{kk}) \right\} \boldsymbol{\gamma}_{ku} \\
&- 2n^{1/2-2v} \hat{\boldsymbol{\Theta}}'^{+}_{pk,n} \left(n^{-(1-v)}\sum_{t=1}^{[nr]} \mathbf{m}_{t,n} z_t - \frac{[nr]}{n} n^{-(1-v)}\sum_{t=1}^{n} \mathbf{m}_{t,n} z_t \right) \\
&+ 2n^{-v} \hat{\boldsymbol{\Theta}}'^{+}_{pk,n} \left(n^{-1/2}\sum_{t=1}^{[nr]} \mathbf{m}_{t,n}\boldsymbol{\varepsilon}'_{k,t} - \frac{[nr]}{n} n^{-1/2}\sum_{t=1}^{n} \mathbf{m}_{t,n}\boldsymbol{\varepsilon}'_{k,t} \right) (\hat{\boldsymbol{\gamma}}_{ku,n} - \boldsymbol{\gamma}_{ku})
\end{aligned}$$

3. A new CUSUM-type testing procedure for the null of cointegration

A note. Scaled partial sum of squared and centered **FM-OLS residuals** (Phillips and Hansen (1990)) from the estimation of a (linear) cointegrating regression model:

Under cointegration:

$$
\begin{aligned}
(1/\sqrt{n})\sum_{t=1}^{[nr]} \hat{v}_{t,p}^{+}(k) &= (1/\sqrt{n})\sum_{t=1}^{[nr]} (z_t^2 - \sigma_n^2) + o_p(1) \\
&= (1/\sqrt{n})\sum_{t=1}^{[nr]} (u_t^2 - \sigma_u^2) - \frac{[nr]}{n}(1/\sqrt{n})\sum_{t=1}^{n} (u_t^2 - \sigma_u^2) \\
&\quad + \boldsymbol{\gamma}'_{ku}\left\{ n^{-1/2}\sum_{t=1}^{[nr]} (\boldsymbol{\varepsilon}_{k,t}\boldsymbol{\varepsilon}'_{k,t} - \boldsymbol{\Sigma}_{kk}) - \frac{[nr]}{n} n^{-1/2}\sum_{t=1}^{n} (\boldsymbol{\varepsilon}_{k,t}\boldsymbol{\varepsilon}'_{k,t} - \boldsymbol{\Sigma}_{kk}) \right\} \boldsymbol{\gamma}_{ku} \\
&\quad \underbrace{- 2\boldsymbol{\gamma}'_{ku}\left\{ n^{-1/2}\sum_{t=1}^{[nr]} (\boldsymbol{\varepsilon}_{k,t}u_t - \boldsymbol{\sigma}_{ku}) - \frac{[nr]}{n} n^{-1/2}\sum_{t=1}^{n} (\boldsymbol{\varepsilon}_{k,t}u_t - \boldsymbol{\sigma}_{ku}) \right\}}
\end{aligned}
$$

The limiting null distribution will differ from that the scaled partial sum of OLS residuals, with the appearance of two additional additive terms that depend on the second order properties of the error driving the integrated regressors and on the *k*-dimensional sample covariance between these and the regression error.

3. A new CUSUM-type testing procedure for the null of cointegration

$$\hat{K}_{n,p}(k) = \frac{1}{\sqrt{n}\hat{\omega}_{v.k,n}(m_n)} \max_{t=1,\ldots,n} \left| \sum_{j=1}^{t} \hat{v}_{j,p}(k) - \hat{\gamma}'_{kv,n}(m_n)(\mathbf{X}_{k,t} - (t/n)\mathbf{X}_{k,n}) \right|$$

Upper quantiles of the null distribution of the CUSUM of squares-type statistic $\hat{K}_{n,p}(k)$ for testing the null of cointegration

Table 1. Case of no deterministic component, $\hat{K}_n(k)$

Sample size, n		Number of integrated regressors, k: $k = 1$	2	3	4	5
$n = 100$	0.90	1.1524	1.1554	1.1601	1.1639	1.1669
	0.95	1.2856	1.2882	1.2848	1.2883	1.2870
	0.975	1.3803	1.3991	1.3909	1.3913	1.3970
	0.99	1.5086	1.5092	1.5142	1.5171	1.5174
$n = 200$	0.90	1.1719	1.1738	1.1746	1.1759	1.1784
	0.95	1.3029	1.3062	1.3072	1.3050	1.3073
	0.975	1.4134	1.4112	1.4137	1.4168	1.4157
	0.99	1.5523	1.5498	1.5527	1.5573	1.5568
$n = 500$	0.90	1.1896	1.1903	1.1910	1.1907	1.1894
	0.95	1.3204	1.3193	1.3202	1.3225	1.3218
	0.975	1.4353	1.4366	1.4371	1.4359	1.4368
	0.99	1.5778	1.5811	1.5787	1.5912	1.5888
$n = 1000$	0.90	1.2007	1.2003	1.1981	1.2011	1.2012
	0.95	1.3327	1.3328	1.3302	1.3306	1.3312
	0.975	1.4501	1.4474	1.4462	1.4461	1.4459
	0.99	1.6045	1.5999	1.5976	1.6040	1.6102
$n = 2000$	0.90	1.2070	1.2067	1.2071	1.2054	1.2061
	0.95	1.3440	1.3458	1.3467	1.3448	1.3445
	0.975	1.4695	1.4653	1.4631	1.4659	1.4669
	0.99	1.6053	1.6048	1.6024	1.6095	1.6108

Table 7. Finite sample-adjusted empirical size at 5% nominal level. Case of no deterministic component, $\hat{K}_n(k)$, with Bartlett kernel and sample size-dependent bandwidth given by $m_n(d) = [d(n/100)^{1/4}]$, $\phi = 0.50$, $\sigma_{k\upsilon} = 0.75$

Sample size, $n = 100$	d	Number of integrated regressors, k: $k = 1$	2	3	4	5
$\alpha = 0.00$	0	0.0516	0.0506	0.0414	0.0412	0.0378
	2	0.0380	0.0388	0.0354	0.0298	0.0276
	4	0.0352	0.0264	0.0230	0.0212	0.0168
	8	0.0278	0.0230	0.0158	0.0184	0.0100
	12	0.0210	0.0200	0.0172	0.0146	0.0114
$\alpha = 0.25$	0	0.0658	0.0728	0.0616	0.0512	0.0454
	2	0.0472	0.0390	0.0338	0.0290	0.0262
	4	0.0426	0.0360	0.0278	0.0258	0.0214
	8	0.0310	0.0228	0.0166	0.0132	0.0092
	12	0.0158	0.0104	0.0126	0.0076	0.0086
$\alpha = 0.50$	0	0.1420	0.1462	0.1218	0.1166	0.1068
	2	0.0574	0.0564	0.0524	0.0400	0.0394
	4	0.0422	0.0334	0.0316	0.0290	0.0248
	8	0.0304	0.0262	0.0230	0.0162	0.0126
	12	0.0212	0.0150	0.0124	0.0106	0.0124
$\alpha = 0.75$	0	0.4394	0.3914	0.3476	0.3106	0.2846
	2	0.1098	0.1034	0.0928	0.0870	0.0696
	4	0.0586	0.0616	0.0538	0.0398	0.0314
	8	0.0292	0.0264	0.0208	0.0144	0.0148
	12	0.0154	0.0102	0.0090	0.0104	0.0074

The simple ad hoc correction for endogeneity works well, and for strongly correlated regression errors and errors driving the integrated regressors the required bandwidth is small to avoid under-rejection.

3. A new CUSUM-type testing procedure for the null of cointegration

Table 14. Finite sample-adjusted empirical size at 5% nominal level.
Case of no deterministic component, $\hat{K}_n(k)$, with Bartlett kernel, sample size-dependent bandwidth given by $m_n(d) = [d(n/100)^{1/4}]$, and $(\alpha,\phi,\sigma_{k\upsilon})$ = (0.50,0.50,0.75)
Heavy-tailed distribution for the cointegrating error term: Student-T, $T(q)$

			Number of integrated regressors, k				
		d	k = 1	2	3	4	5
q = 4	n = 100	0	0.1474	0.1308	0.1240	0.1030	0.0858
		2	0.0506	0.0526	0.0444	0.0350	0.0274
		4	0.0356	0.0458	0.0320	0.0228	0.0224
		8	0.0282	0.0226	0.0150	0.0122	0.0086
		12	0.0148	0.0160	0.0118	0.0098	0.0100
	n = 200	0	0.1594	0.1712	0.1544	0.1398	0.1428
		2	0.0680	0.0718	0.0620	0.0538	0.0472
		4	0.0458	0.0500	0.0368	0.0334	0.0348
		8	0.0360	0.0374	0.0276	0.0262	0.0218
		12	0.0216	0.0230	0.0198	0.0142	0.0128
	n = 500	0	0.1850	0.1858	0.1870	0.1718	0.1670
		2	0.0676	0.0766	0.0826	0.0628	0.0626
		4	0.0492	0.0574	0.0486	0.0478	0.0414
		8	0.0464	0.0464	0.0390	0.0346	0.0286
		12	0.0308	0.0354	0.0304	0.0200	0.0258
q = 3	n = 100	0	0.1454	0.1486	0.1220	0.0996	0.0980
		2	0.0488	0.0564	0.0418	0.0392	0.0310
		4	0.0352	0.0416	0.0290	0.0256	0.0186
		8	0.0246	0.0288	0.0146	0.0166	0.0124
		12	0.0092	0.0102	0.0096	0.0052	0.0070
	n = 200	0	0.1590	0.1788	0.1406	0.1242	0.1238
		2	0.0578	0.0700	0.0566	0.0510	0.0448
		4	0.0506	0.0608	0.0420	0.0310	0.0312
		8	0.0290	0.0326	0.0274	0.0192	0.0178
		12	0.0214	0.0258	0.0188	0.0108	0.0132
	n = 500	0	0.1656	0.1780	0.1486	0.1424	0.1358
		2	0.0624	0.0694	0.0658	0.0542	0.0540
		4	0.0444	0.0526	0.0418	0.0366	0.0324
		8	0.0324	0.0424	0.0270	0.0282	0.0252
		12	0.0318	0.0352	0.0298	0.0212	0.0220

			Number of integrated regressors, k				
		d	k = 1	2	3	4	5
q = 2	n = 100	0	0.1118	0.1394	0.0990	0.0862	0.0720
		2	0.0324	0.0508	0.0294	0.0208	0.0212
		4	0.0188	0.0362	0.0180	0.0150	0.0148
		8	0.0172	0.0270	0.0154	0.0158	0.0116
		12	0.0094	0.0112	0.0088	0.0104	0.0122
	n = 200	0	0.1242	0.1488	0.1124	0.1020	0.0902
		2	0.0418	0.0630	0.0426	0.0344	0.0340
		4	0.0206	0.0366	0.0268	0.0202	0.0186
		8	0.0204	0.0326	0.0148	0.0126	0.0120
		12	0.0150	0.0236	0.0116	0.0104	0.0098
	n = 500	0	0.1342	0.1980	0.1268	0.1340	0.1250
		2	0.0336	0.0636	0.0314	0.0270	0.0296
		4	0.0234	0.0448	0.0204	0.0190	0.0208
		8	0.0150	0.0376	0.0140	0.0120	0.0128
		12	0.0152	0.0312	0.0160	0.0128	0.0096

3. A new CUSUM-type testing procedure for the null of cointegration

Table 15. Finite sample-adjusted empirical size at 5% nominal level.
Case of no deterministic component, $\hat{\mathrm{K}}_n(k)$, with Bartlett kernel, sample size-dependent bandwidth given by $m_n(d)=[d(n/100)^{1/4}]$, and $(\alpha,\phi,\sigma_{k\upsilon})$ = (0.00,0.00,0.00). **15.1.** N(0,1)-GARCH(1,1), $\upsilon_t = z_t h_t$, $h_t^2 = \alpha_0 + \alpha_1 \upsilon_{t-1}^2 + \beta_1 h_{t-1}^2$

			Number of integrated regressors, k				
(α_0, α_1, β_1)	Sample size, n	d	k = 1	2	3	4	5
(0.10, 0.05, 0.65)	n = 100	0	0.0916	0.0920	0.0900	0.0906	0.0850
		2	0.0642	0.0604	0.0552	0.0518	0.0460
		4	0.0458	0.0394	0.0376	0.0368	0.0308
		8	0.0286	0.0256	0.0204	0.0202	0.0170
		12	0.0186	0.0156	0.0130	0.0112	0.0094
	n = 200	0	0.1082	0.1118	0.1050	0.1082	0.1050
		2	0.0798	0.0788	0.0724	0.0718	0.0708
		4	0.0586	0.0536	0.0498	0.0498	0.0486
		8	0.0410	0.0366	0.0324	0.0260	0.0242
		12	0.0340	0.0258	0.0220	0.0232	0.0218
	n = 500	0	0.1112	0.1076	0.1066	0.1088	0.1072
		2	0.0834	0.0838	0.0838	0.0838	0.0798
		4	0.0610	0.0624	0.0596	0.0570	0.0572
		8	0.0488	0.0470	0.0434	0.0436	0.0428
		12	0.0374	0.0334	0.0304	0.0272	0.0256
(0.10, 0.05, 0.75)	n = 100	0	0.1196	0.1182	0.1200	0.1148	0.1142
		2	0.0778	0.0738	0.0712	0.0662	0.0602
		4	0.0624	0.0526	0.0488	0.0450	0.0360
		8	0.0330	0.0268	0.0236	0.0214	0.0206
		12	0.0184	0.0176	0.0144	0.0118	0.0118
	n = 200	0	0.1430	0.1416	0.1332	0.1338	0.1342
		2	0.1024	0.0980	0.0946	0.0894	0.0908
		4	0.0770	0.0748	0.0722	0.0690	0.0650
		8	0.0408	0.0400	0.0350	0.0316	0.0302
		12	0.0314	0.0312	0.0278	0.0238	0.0200
	n = 500	0	0.1528	0.1480	0.1488	0.1510	0.1554
		2	0.1236	0.1208	0.1198	0.1160	0.1162
		4	0.0866	0.0858	0.0838	0.0832	0.0822
		8	0.0600	0.0548	0.0532	0.0516	0.0496
		12	0.0460	0.0432	0.0426	0.0368	0.0372
(0.10, 0.05, 0.90)	n = 100	0	0.4002	0.3874	0.3738	0.3660	0.3622
		2	0.3316	0.3114	0.2878	0.2736	0.2550
		4	0.2602	0.2388	0.2188	0.2002	0.1836
		8	0.1476	0.1168	0.0968	0.0828	0.0744
		12	0.0786	0.0690	0.0540	0.0488	0.0394
	n = 200	0	0.4284	0.4220	0.4260	0.4158	0.4066
		2	0.3518	0.3384	0.3272	0.3148	0.3068
		4	0.2922	0.2794	0.2678	0.2536	0.2450
		8	0.1864	0.1708	0.1552	0.1448	0.1334
		12	0.1218	0.1072	0.0890	0.0854	0.0752
	n = 500	0	0.5066	0.5038	0.4982	0.4938	0.4900
		2	0.4264	0.4282	0.4234	0.4182	0.4114
		4	0.3358	0.3292	0.3216	0.3180	0.3100
		8	0.2198	0.2074	0.2022	0.1964	0.1880
		12	0.1464	0.1424	0.1348	0.1290	0.1234

3. A new CUSUM-type testing procedure for the null of cointegration

Table 15. Finite sample-adjusted empirical size at 5% nominal level.

Case of no deterministic component, $\hat{K}_n(k)$, with Bartlett kernel, sample size-dependent bandwidth given by $m_n(d) = [d(n/100)^{1/4}]$, and $(\alpha,\phi,\sigma_{k\upsilon}) = (0.00,0.00,0.00)$. **15.2.** T($q$)-GARCH(1,1), $\upsilon_t = z_t h_t$, $h_t^2 = \alpha_0 + \alpha_1 \upsilon_{t-1}^2 + \beta_1 h_{t-1}^2$, $q = 5$

			Number of integrated regressors, k				
(α_0, α_1, β_1)	Sample size, n	d	$k = 1$	2	3	4	5
(0.10, 0.05, 0.65)	$n = 100$	0	0.1070	0.0998	0.0984	0.0958	0.1044
		2	0.0574	0.0630	0.0560	0.0498	0.0542
		4	0.0404	0.0366	0.0302	0.0290	0.0268
		8	0.0194	0.0166	0.0164	0.0128	0.0128
		12	0.0146	0.0112	0.0126	0.0134	0.0108
	$n = 200$	0	0.1422	0.1480	0.1438	0.1400	0.1372
		2	0.0862	0.0824	0.0784	0.0790	0.0722
		4	0.0554	0.0530	0.0490	0.0488	0.0426
		8	0.0350	0.0232	0.0320	0.0230	0.0214
		12	0.0220	0.0202	0.0194	0.0154	0.0170
	$n = 500$	0	0.1488	0.1460	0.1484	0.1514	0.1484
		2	0.0808	0.0906	0.0898	0.0882	0.0878
		4	0.0618	0.0556	0.0576	0.0610	0.0538
		8	0.0372	0.0354	0.0324	0.0270	0.0332
		12	0.0274	0.0252	0.0248	0.0222	0.0252
(0.10, 0.05, 0.75)	$n = 100$	0	0.1398	0.1400	0.1406	0.1436	0.1392
		2	0.0920	0.0862	0.0892	0.0808	0.0684
		4	0.0584	0.0502	0.0500	0.0458	0.0422
		8	0.0274	0.0288	0.0188	0.0228	0.0182
		12	0.0176	0.0140	0.0182	0.0132	0.0126
	$n = 200$	0	0.1762	0.1752	0.1608	0.1736	0.1594
		2	0.1170	0.1148	0.1170	0.1082	0.0988
		4	0.0792	0.0776	0.0782	0.0742	0.0718
		8	0.0432	0.0394	0.0390	0.0326	0.0338
		12	0.0254	0.0210	0.0176	0.0152	0.0146
	$n = 500$	0	0.2106	0.2172	0.2148	0.2050	0.2224
		2	0.1532	0.1536	0.1484	0.1526	0.1556
		4	0.0918	0.0922	0.0908	0.0908	0.0896
		8	0.0542	0.0516	0.0546	0.0544	0.0434
		12	0.0410	0.0410	0.0354	0.0300	0.0324
(0.10, 0.05, 0.90)	$n = 100$	0	0.4720	0.4582	0.4638	0.4612	0.4452
		2	0.3800	0.3638	0.3450	0.3202	0.3136
		4	0.2982	0.2804	0.2618	0.2460	0.2268
		8	0.1802	0.1524	0.1272	0.1098	0.0948
		12	0.1108	0.0856	0.0710	0.0660	0.0498
	$n = 200$	0	0.5750	0.5794	0.5780	0.5820	0.5778
		2	0.5204	0.5036	0.4924	0.4796	0.4750
		4	0.4586	0.4492	0.4414	0.4180	0.4070
		8	0.2692	0.2486	0.2452	0.2214	0.2082
		12	0.1946	0.1754	0.1534	0.1428	0.1292
	$n = 500$	0	0.7632	0.7474	0.7516	0.7460	0.7452
		2	0.6774	0.6712	0.6676	0.6700	0.6640
		4	0.5744	0.5680	0.5644	0.5682	0.5600
		8	0.3900	0.3844	0.3762	0.3792	0.3706
		12	0.2802	0.2746	0.2704	0.2582	0.2476

4. The effects of highly autocorrelated regression errors and test consistency

4.1. Highly autocorrelated regression errors. From part (a) of Assumption L:

$$u_t = \alpha u_{t-1} + \upsilon_t, \quad \alpha = \alpha_n = 1 - \lambda n^{-1}, \quad \lambda \geq 0$$

$$n^{-1/2} u_{[nr]} \Rightarrow \omega_\upsilon J_\lambda(r) \qquad \textbf{(a)}\; u_0 = o_p(n^{-1/2}) \quad \text{(Phillips (1987))}$$

$$\Rightarrow \omega_\upsilon (e^{-\lambda r}\zeta_\lambda + J_\lambda(r)) = \omega_\upsilon M_\lambda(r) \quad \textbf{(b)}\; u_0 = \textstyle\sum_{s=0}^{\infty} \alpha^s \upsilon_{-s} \quad \text{(Müller (2005))},\; \zeta_\lambda \sim N(0, 1/2\lambda)$$

Proposition 4.1:

- $(1/\sqrt{n})\sum_{t=1}^{[nr]} v_t = O_p(n\sqrt{n})$
- $(1/n^2)\sum_{t=1}^{[nr]} v_t \Rightarrow (\omega_\upsilon^2/\lambda)\left\{\int_0^r J_\lambda(s)dW_\upsilon(s) - r\int_0^1 J_\lambda(s)dW_\upsilon(s)\right\} - \omega_\upsilon^2(J_\lambda^2(r) - rJ_\lambda^2(1))$ **(a)**

$$\Rightarrow (\omega_\upsilon^2/\lambda)\left\{\int_0^r M_\lambda(s)dW_\upsilon(s) - r\int_0^1 M_\lambda(s)dW_\upsilon(s)\right\} + (1/\lambda)(1-r)\omega_\upsilon^2\zeta_\lambda^2 - \omega_\upsilon^2(M_\lambda^2(r) - rM_\lambda^2(1)) \quad \textbf{(b)}$$

4.2. Consistency

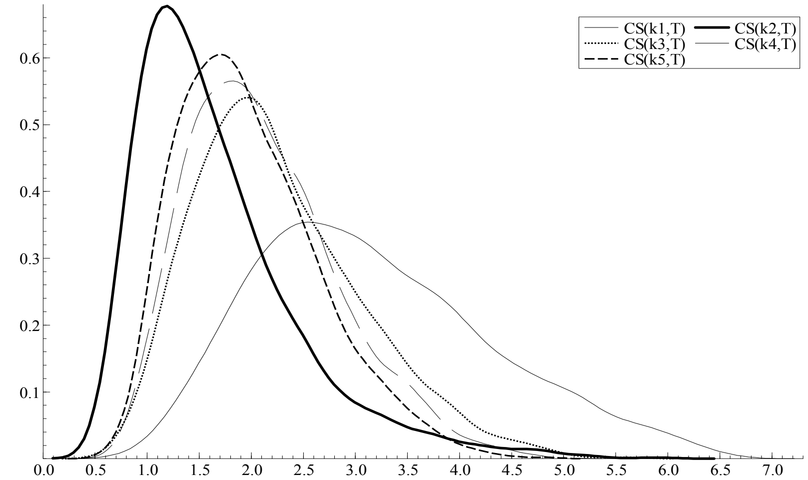
Proposition 4.2. When λ = 0, with $\hat{u}_{t,p}(k) = \hat{\kappa}'_{k,n}\eta_{t,p}, \hat{\kappa}_{k,n} = (1, -\hat{\beta}'_{k,n})'$:

- $(1/\sqrt{n})\sum_{t=1}^{[nr]} \hat{v}_{t,p}(k) = O_p(n\sqrt{n})$
- $\hat{\omega}^2_{v.k,n}(m_n) = O_p(m_n n^2), \hat{\omega}_{kv,n}(m_n) = O_p(m_n\sqrt{n})$
- $\hat{K}_{n,p}(k) = O_p(\sqrt{n/m_n})$

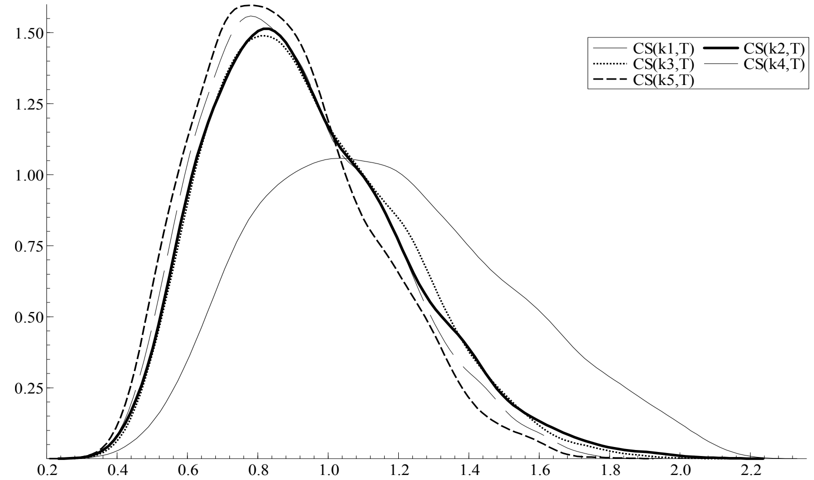
5. Finite sample power results

Nonparametric kernel estimation of the density function of the CUSUM-of squares test statistics computed under the alternative of no cointegration

Case A. No deterministic component, with sample size $n = 200$, $(\sigma_{k\upsilon}, \phi) = (0.75, 0.50)$, and sample size-dependent deterministic bandwidth $m_n(d) = [d/(n/100)^{1/4}]$

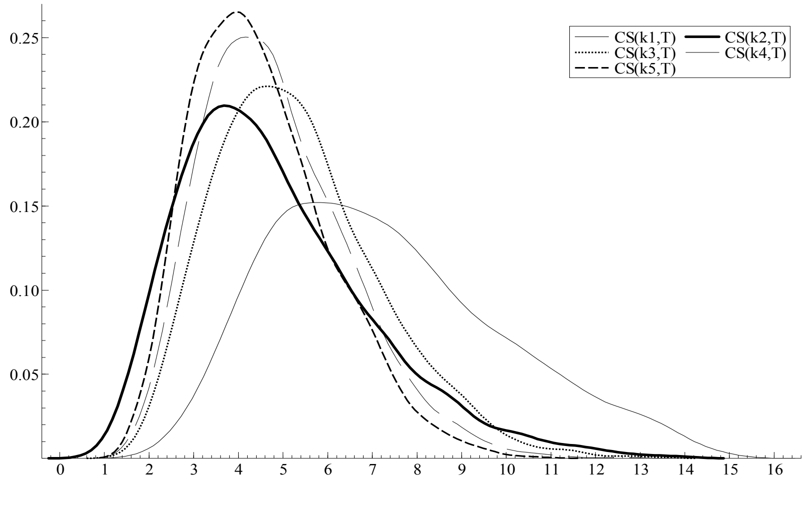


$d = 0$

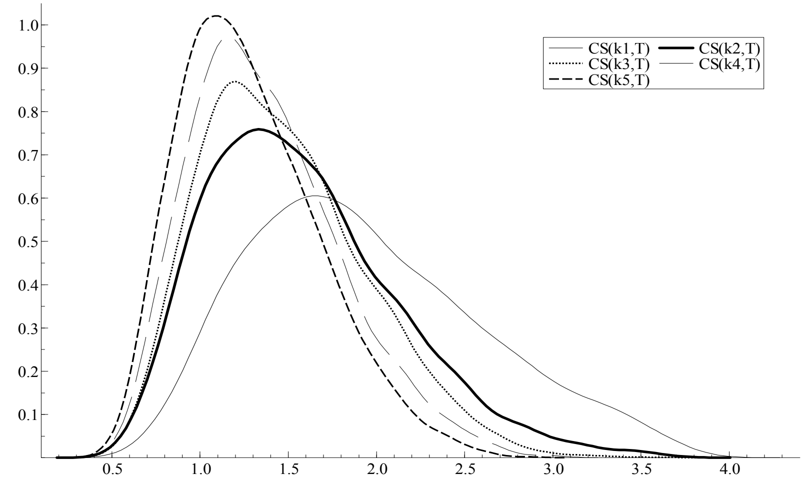


$d = 8$

Case B. No deterministic component, with sample size $n = 1000$, $(\sigma_{k\upsilon}, \phi) = (0.75, 0.50)$, and sample size-dependent deterministic bandwidth $m_n(d) = [d/(n/100)^{1/4}]$

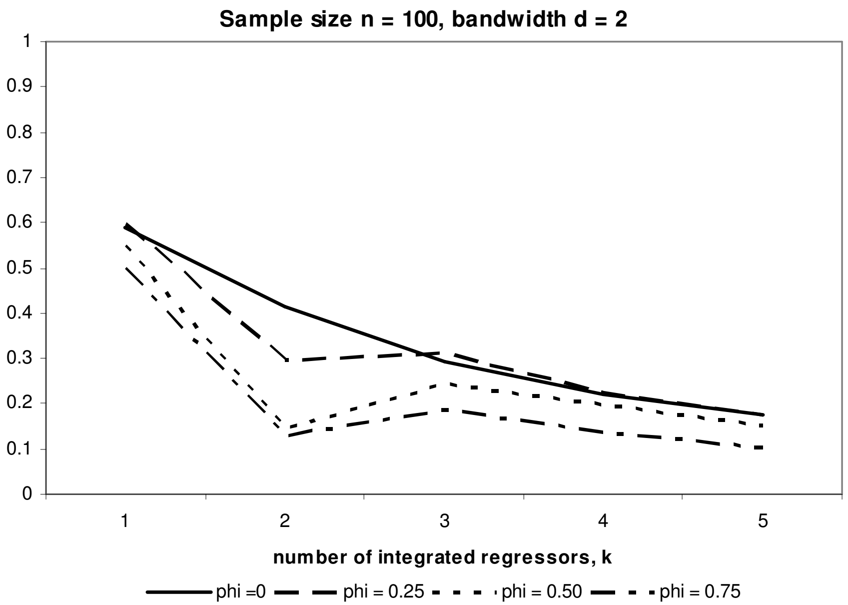


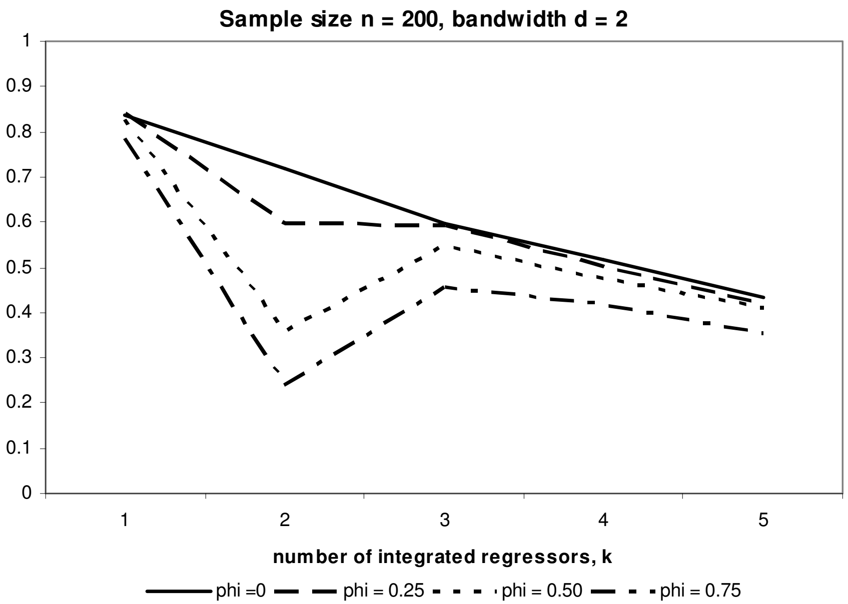
$d = 0$

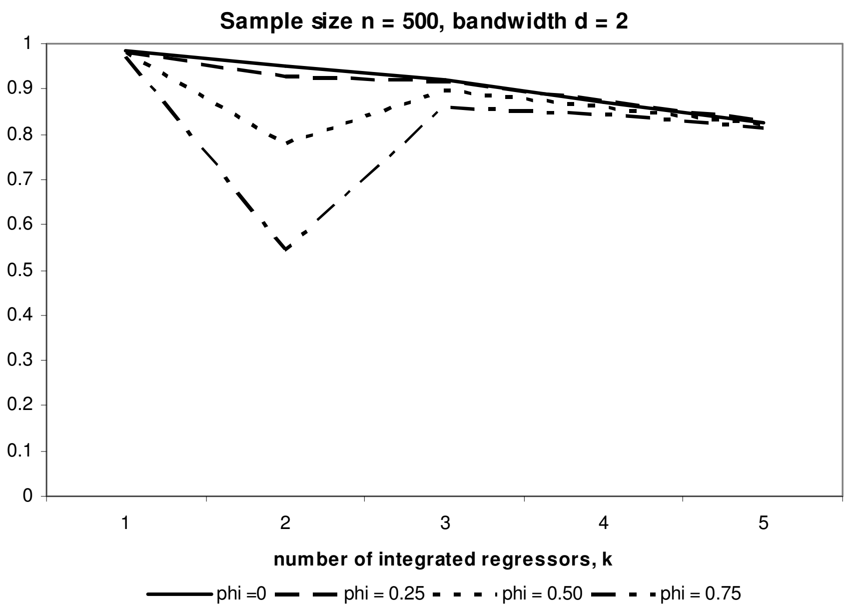


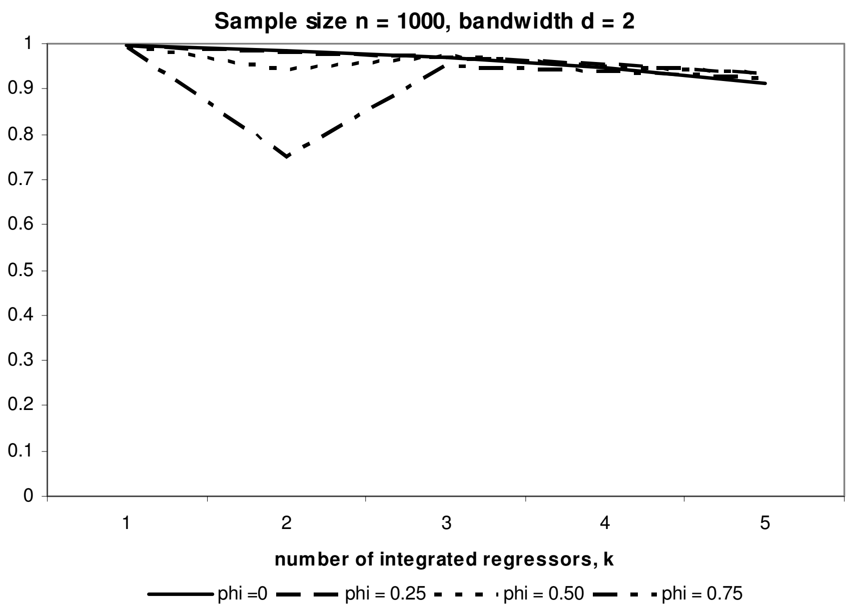
$d = 8$

5. Finite sample power results

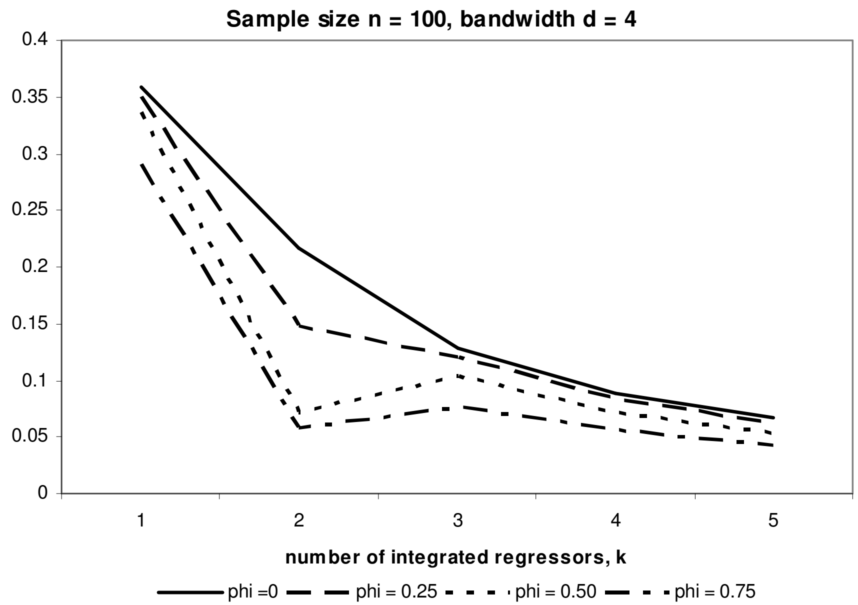


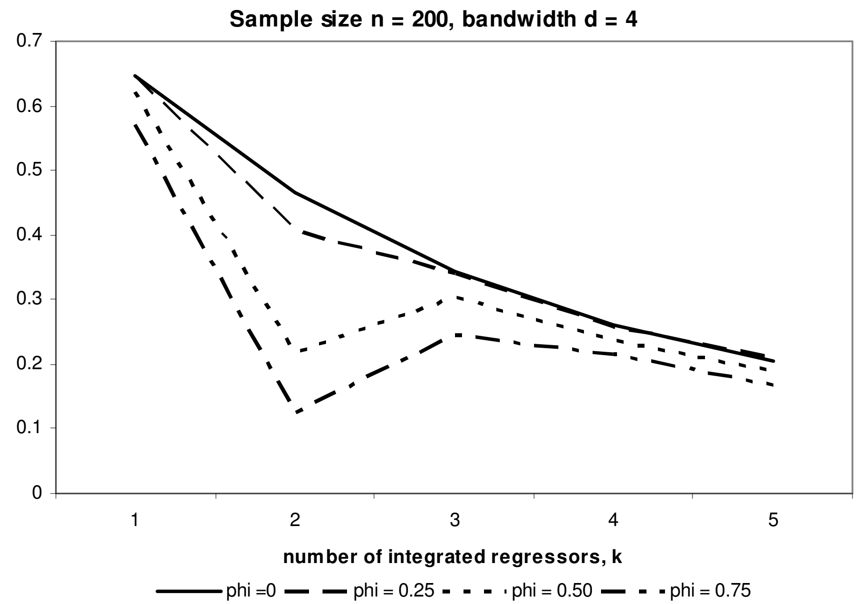


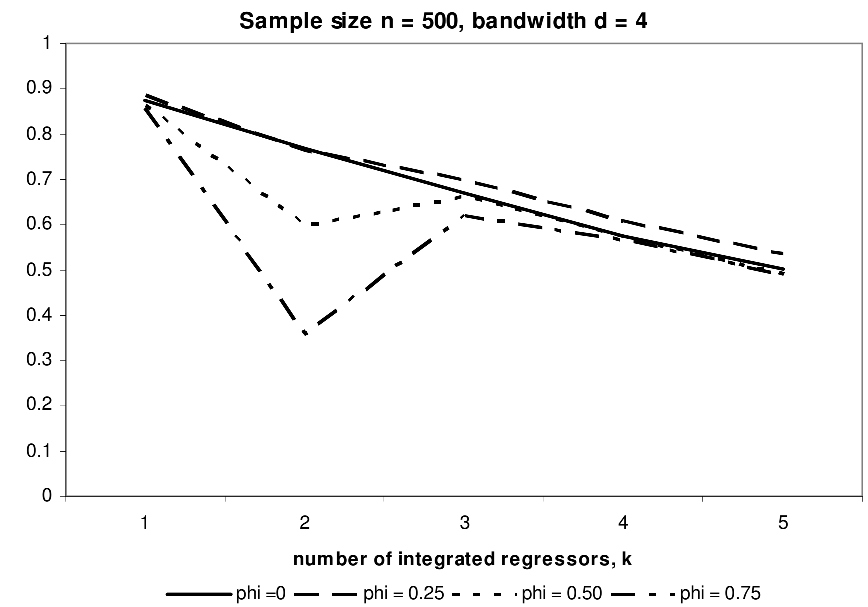


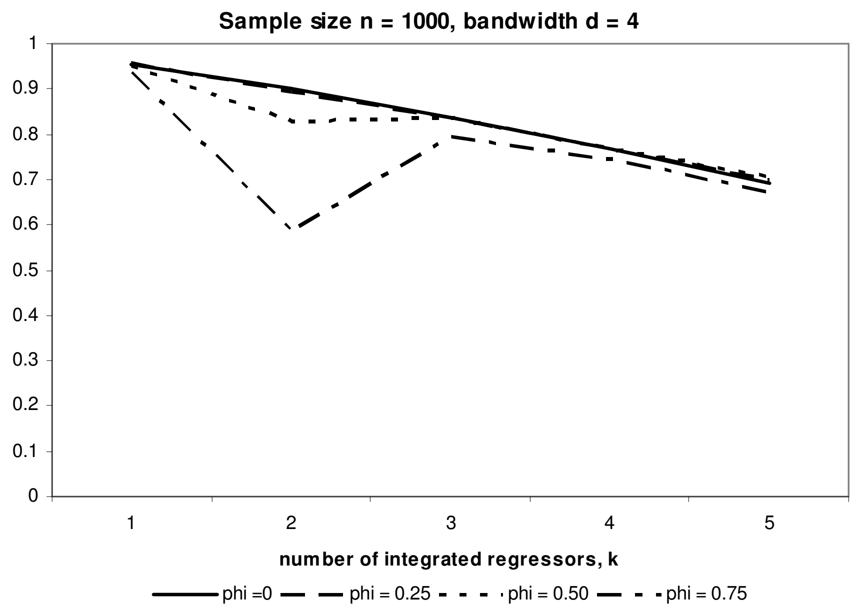


5. Finite sample power results









Some concluding comments:

The proposed fluctuation-type test for the null of cointegration in single-equation cointegrating regression models have the following interesting advantages:

- It is relatively simple to compute and only makes use of the OLS residuals from the estimation of the cointegrating regression, and exploits their information content as consistent estimators of the regression errors.
- The limiting null distribution does not depend on any particular feature of the specified model, whenever all the model parameters are consistently estimated. This model-free limiting null distribution has the following appealing implications:
  (a) Simple to use in practice with a single set of critical values
  (b) Prevents for the inconvenients caused by: subcointegration, spurious cointegration evidence caused by deterministically trending integrated regressors (see Hassler (2001)), …
- As usual, the consistency rate under no cointegration mainly depends on the bandwidth used to estimate the long-run variances and covariances, but the numerical results indicate that when based on the squared residuals relatively small values of the bandwidth parameter are required.
- The numerical results on size and power in finite samples show that, as compared with related existing testing procedures, this new testing procedure displays good properties even in small samples and even under some situations violating the technical assumptions required to develop the asymptotics.

Thank you for your attention!!!!

… and any comment or suggestion is welcome